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S.2404 If $M \in Int(\Delta ABC), d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$ then

$$\frac{a^m b}{h_a h_b^m} + \frac{b^m c}{h_b h_c^m} + \frac{c^m a}{h_c h_a^m} \geq 2^{m+1}\left(\sqrt{3}\right)^{1-m}$$

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We have $ah_a = bh_b = ch_c = 2F$. Using $AM - GM$ inequality and

Carlitz's inequality $(abc)^{2/3} \geq \frac{4}{3}\sqrt{3}F$, it follows that:

$$\frac{a^m b}{h_a h_b^m} + \frac{b^m c}{h_b h_c^m} + \frac{c^m a}{h_c h_a^m} = \frac{a^{m+1} b^{m+1}}{a h_a b^m h_b^m} + \frac{b^{m+1} c^{m+1}}{b h_b c^m h_c^m} + \frac{c^{m+1} a^{m+1}}{c h_c a^m h_a^m} =$$

$$= \frac{(ab)^{m+1}}{(2F)^{m+1}} + \frac{(bc)^{m+1}}{(2F)^{m+1}} + \frac{(ca)^{m+1}}{(2F)^{m+1}} \geq \frac{3}{2^{m+1}F^{m+1}}\left(a^2b^2c^2\right)^{\frac{m+1}{3}} \geq$$

$$\geq \frac{3}{2^{m+1}F^{m+1}}\left(\frac{4}{3}\sqrt{3}F\right)^{m+1} = 2^{m+1}\left(\sqrt{3}\right)^{1-m}$$

Equality holds if and only if ΔABC is equilateral.