ROMANIAN MATHEMATICAL MAGAZINE

S.2405 If $m\ge 0$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}\cdot h_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}\cdot h_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}\cdot h_c^m}\ge 2\cdot\left(\sqrt{3}\right)^{1-m}\cdot F$$

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We have $ah_a=bh_b=ch_c=2F$. By Power Mean inequality we obtain

$$\left(\frac{a^{m+1}+b^{m+1}+c^{m+1}}{3}\right)^{\frac{1}{m+1}}\ge\frac{a+b+c}{3}\Leftrightarrow$$

$$3^m\left(a^{m+1}+b^{m+1}+c^{m+1}\right)\ge(a+b+c)^{m+1}\quad(1)$$

Applying inequality (1) and Tsintsinfas' inequality $\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\ge 2\sqrt{3}F$,

it follows that

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}\cdot h_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}\cdot h_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}\cdot h_c^m}=$$

$$=\frac{x^{m+1}\cdot a^{2m+2}}{(y+z)^{m+1}\cdot a^m h_a^m}+\frac{y^{m+1}\cdot b^{2m+2}}{(z+x)^{m+1}\cdot b^m h_b^m}+\frac{z^{m+1}\cdot c^{2m+2}}{(x+y)^{m+1}\cdot c^m h_c^m}=$$

$$=\frac{1}{2^mF^m}\left(\left(\frac{xa^2}{y+z}\right)^{m+1}+\left(\frac{yb^2}{z+x}\right)^{m+1}+\left(\frac{zc^2}{x+y}\right)^{m+1}\right)\ge$$

$$\ge\frac{1}{2^mF^m}\cdot\frac{1}{3^m}\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)^{m+1}\ge$$

$$\ge\frac{1}{2^mF^m}\cdot\frac{1}{3^m}\left(2\sqrt{3}F\right)^{m+1}=2\cdot\left(\sqrt{3}\right)^{1-m}\cdot F.$$

Equality holds if and only if ΔABC is equilateral and $x=y=z$.