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S.2407 In ΔABC, g_a, g_b, g_c Gergonne's cevians, the following relationship holds:

$$\frac{a^3\cdot g_a^2}{\sqrt{g_bg_c}}+\frac{b^3\cdot g_b^2}{\sqrt{g_cg_a}}+\frac{c^3\cdot g_c^2}{\sqrt{g_ag_b}}\geq 8\sqrt{3}\cdot F^2$$

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Since $g_a \geq h_a$, we have $a^3\cdot g_a^2 \geq a^3\cdot h_ag_a = a^2\cdot g_a(ah_a) = 2F\cdot a^2g_a$.

Applying $AM-GM$ inequality and Carlitz's inequality $(abc)^{2/3}\geq \frac{4}{3}\sqrt{3}\cdot F$, it follows that:

$$\frac{a^3\cdot g_a^2}{\sqrt{g_bg_c}}+\frac{b^3\cdot g_b^2}{\sqrt{g_cg_a}}+\frac{c^3\cdot g_c^2}{\sqrt{g_ag_b}}\geq 2F\left(\frac{a^2g_a}{\sqrt{g_bg_c}}+\frac{b^2g_b}{\sqrt{g_cg_a}}+\frac{c^2g_c}{\sqrt{g_ag_b}}\right)\geq$$

$$6F\left(\frac{a^2g_a}{\sqrt{g_bg_c}}\cdot\frac{b^2g_b}{\sqrt{g_cg_a}}\cdot\frac{c^2g_c}{\sqrt{g_ag_b}}\right)^{\frac{1}{3}}=6F(abc)^{\frac{2}{3}}\geq 6F\left(\frac{4}{3}\sqrt{3}\cdot F\right)=8\sqrt{3}\cdot F^2$$

Equality holds if and only if ΔABC is equilateral.