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S.2408 If $x,y,z>0$, then in ΔABC holds

$$\frac{x}{h_a h_b\sqrt{yz}}+\frac{y}{h_b h_c\sqrt{zx}}+\frac{z}{h_c h_a\sqrt{xy}}\geq\frac{\sqrt{3}}{F}$$

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We have $ah_a=bh_b=ch_c=2F$. Using $AM-GM$ inequality and

Carlitz's inequality $(abc)^{2/3}\geq\frac{4}{3}\sqrt{3}F$, it follows that:

$$\frac{x}{h_a h_b\sqrt{yz}}+\frac{y}{h_b h_c\sqrt{zx}}+\frac{z}{h_c h_a\sqrt{xy}}=\frac{xab}{4F^2\sqrt{yz}}+\frac{ybc}{4F^2\sqrt{zx}}+\frac{zca}{4F^2\sqrt{xy}}\geq$$

$$\geq\frac{3}{4F^2}\left(\frac{x}{\sqrt{yz}}\cdot\frac{y}{\sqrt{zx}}\cdot\frac{z}{\sqrt{xy}}a^2b^2c^2\right)^{1/3}\geq\frac{3}{4F^2}\left(\frac{4}{3}\sqrt{3}F\right)=\frac{\sqrt{3}}{F}.$$

Equality holds if and only if $x=y=z$ and ΔABC is equilateral.