ROMANIAN MATHEMATICAL MAGAZINE

S.2409 If $x \geq 0$ and in ΔABC, $A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)$ such that $BA_1 = xA_1C,\ CB_1 = xB_1A,\ AC_1 = xC_1B$, and

$a_1 = B_1C_1,\ b_1 = C_1A_1,\ c_1 = A_1B_1$, then:

$$a_1b_1 + b_1c_1 + c_1a_1 \geq \frac{4x \cdot \sqrt{3}}{(x+1)^2} \cdot F$$

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By $BA_1 = xA_1C, BA_1 + A_1C = a$, we get $BA_1 = \frac{ax}{x+1}, A_1C = \frac{a}{x+1}$,

and similar $CB_1 = \frac{bx}{x+1}, B_1A = \frac{b}{x+1}, AC_1 = \frac{cx}{x+1}, C_1B = \frac{c}{x+1}$.

We obtain $[AB_1C_1] = \frac{AB_1AC_1\sin A}{2} = \frac{bcx\sin A}{2(x+1)^2} = \frac{x}{(x+1)^2} \cdot F$. It follows that:

$$[A_1B_1C_1] = [ABC] - [AB_1C_1] - [BC_1A_1] - [CA_1B_1] =$$

$$= F - \frac{3x}{(x+1)^2} \cdot F = \frac{x^2 - x + 1}{(x+1)^2} \cdot F.$$

By $AM - GM$ we have $x^2 - x + 1 = x^2 + 1 - x \geq 2x - x = x$. It results that:

$$[A_1B_1C_1] \geq \frac{x}{(x+1)^2} \cdot F \quad (1)$$

Applying Gordon's inequality for triangle $A_1B_1C_1$:

$a_1b_1 + b_1c_1 + c_1a_1 \geq 4\sqrt{3}[A_1B_1C_1]$ and inequality (1), it follows that

$$a_1b_1 + b_1c_1 + c_1a_1 \geq 4\sqrt{3}[A_1B_1C_1] \geq \frac{4x \cdot \sqrt{3}}{(x+1)^2} \cdot F.$$

Equality holds if and only if $x = 1, a = b = c$, that is if and only if $\Delta A_1B_1C_1$ is the median triangle of echilateral ΔABC.