ROMANIAN MATHEMATICAL MAGAZINE

S.2411 If $m \geq 0$ and $x, y > 0$ then in ΔABC holds:

$$\frac{a^{m+1}\cdot b}{(ax+by)^m\cdot h_b^m}+\frac{b^{m+1}\cdot c}{(bx+cy)^m\cdot h_c^m}+\frac{c^{m+1}\cdot a}{(cx+ay)^m\cdot h_a^m}\geq\frac{4\left(\sqrt{3}\right)^{m+1}\cdot r}{(x+y)^m\cdot s^{m-1}}$$

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We have $ah_a = bh_b = ch_c = 2F$ and $F = sr$.

Applying Radon's inequality and Gordon's inequality $ab + bc + ca \geq 4\sqrt{3}F$, we obtain:

$$\frac{a^{m+1}\cdot b}{(ax+by)^m\cdot h_b^m}+\frac{b^{m+1}\cdot c}{(bx+cy)^m\cdot h_c^m}+\frac{c^{m+1}\cdot a}{(cx+ay)^m\cdot h_a^m}=$$

$$=\frac{(ab)^{m+1}}{(ax+by)^m\cdot(2F)^m}+\frac{(bc)^{m+1}}{(bx+cy)^m\cdot(2F)^m}+\frac{(ca)^{m+1}}{(cx+ay)^m\cdot(2F)^m}\geq$$

$$\geq\frac{(ab+bc+ca)^{m+1}}{2^mF^m\left(x(a+b+c)+y(a+b+c)\right)^m}\geq$$

$$\geq\frac{4^{m+1}\left(\sqrt{3}\right)^{m+1}F^{m+1}}{2^mF^m2^ms^m(x+y)^m}=\frac{4\left(\sqrt{3}\right)^{m+1}\cdot r}{(x+y)^m\cdot s^{m-1}}.$$

Equality holds if and only if ΔABC is equilateral.