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S.2414 If $t>0$, then in ΔABC holds:

$$\left(\left(\frac{a^3}{bR+cr}\right)^2+t^2\right)\left(\left(\frac{b^3}{cR+ar}\right)^2+t^2\right)\left(\left(\frac{c^3}{aR+br}\right)^2+t^2\right)\geq\frac{36t^4}{(R+r)^2}F^2.$$

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It is known the inequality of Arkady Alt (see for example $[1]$):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2 \qquad (1)$$

with equality if and only if $x=y=z, t=x\sqrt{2}$.

Using (1), it follows that

$$\left(\left(\frac{a^3}{bR+cr}\right)^2+t^2\right)\left(\left(\frac{b^3}{cR+ar}\right)^2+t^2\right)\left(\left(\frac{c^3}{aR+br}\right)^2+t^2\right)\geq$$

$$\geq\frac{3}{4}t^4\left(\frac{a^3}{bR+cr}+\frac{b^3}{cR+ar}+\frac{c^3}{aR+br}\right)^2.$$

It remains to prove that

$$\frac{3}{4}t^4\left(\frac{a^3}{bR+cr}+\frac{b^3}{cR+ar}+\frac{c^3}{aR+br}\right)^2\geq\frac{36t^4}{(R+r)^2}F^2$$

that is

$$\frac{a^3}{bR+cr}+\frac{b^3}{cR+ar}+\frac{c^3}{aR+br}\geq\frac{4\sqrt{3}F}{R+r} \qquad (2)$$

Applyind Bergström's inequality, Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\geq4\sqrt{3}F$ and the well known inequality $ab+bc+ca\leq a^2+b^2+c^2$, it follows that

$$\frac{a^3}{bR+cr}+\frac{b^3}{cR+ar}+\frac{c^3}{aR+br}=\frac{a^4}{abR+car}+\frac{b^4}{bcR+abr}+\frac{c^4}{caR+bcr}\geq$$

$$\geq\frac{(a^2+b^2+c^2)^2}{R(ab+bc+ca)+r(ab+bc+ca)}\geq\frac{(a^2+b^2+c^2)^2}{(R+r)(a^2+b^2+c^2)}\geq\frac{4\sqrt{3}F}{R+r},$$

hence the inequality (2) is true.

Equality holds if and only if ΔABC is equilateral and $t=\frac{a^3\sqrt{2}}{\frac{a^2}{\sqrt{3}}+\frac{a^2}{2\sqrt{3}}}=a\sqrt{\frac{2}{3}}$.

$[1]$ D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă la APMO 2004*, Recreații Matematice nr. 1/2024

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ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.