ROMANIAN MATHEMATICAL MAGAZINE

S.2416 If $u,v>0$ and $M\in Int(\Delta ABC)$, $x_a=d(M,BC)$, $x_b=d(M,CA)$, $x_c=d(M,AB)$ then:

$$\frac{a}{uh_a+vx_a}+\frac{b}{uh_b+vx_b}+\frac{c}{uh_c+vx_c}\geq\frac{6\sqrt{3}}{3u+v}$$

Proposed by D.M. Bătinețu–Giurgiu – Romania

Solution by Titu Zvonaru-Romania

Let F be the area of ΔABC. We have $ah_a=bh_b=ch_c=2F$, $ax_a+bx_b+cx_c=2F$.

By $AM-GM$ inequality and Carlitz's inequality $(abc)^{2/3}\geq\frac{4}{3}F\sqrt{3}$ we obtain

$$(a+b+c)^2\geq 9(abc)^{\frac{2}{3}}\geq 12F\sqrt{3}.$$

Using Bergström inequality, it follows that

$$\frac{a}{uh_a+vx_a}+\frac{b}{uh_b+vx_b}+\frac{c}{uh_c+vx_c}\geq$$

$$\geq\frac{(a+b+c)^2}{u(ah_a+bh_b+ch_c)+v(ax_a+bx_b+cx_c)}\geq\frac{12F\sqrt{3}}{(6u+2v)F}=\frac{6\sqrt{3}}{3u+v}$$