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S.2418 If $x > 0$, then in ΔABC holds:

$$\left(\frac{m_a^{2x+4}}{(m_bR+m_cr)^{2x}}+1\right)\left(\frac{m_b^{2x+4}}{(m_cR+m_ar)^{2x}}+1\right)\left(\frac{m_c^{2x+4}}{(m_aR+m_br)^{2x}}+1\right)\ge\frac{81}{4(R+r)^{2x}}\cdot F^2$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$, Radon's inequality, the known inequality $x^2+y^2+z^2\ge xy+yz+zx$ and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\ge 4\sqrt{3}F$,it follows that:

$$\left(\frac{m_a^{2x+4}}{(m_bR+m_cr)^{2x}}+1\right)\left(\frac{m_b^{2x+4}}{(m_cR+m_ar)^{2x}}+1\right)\left(\frac{m_c^{2x+4}}{(m_aR+m_br)^{2x}}+1\right)\ge$$

$$\ge\frac{3}{4}\left(\frac{m_a^{x+2}}{(m_bR+m_cr)^x}+\frac{m_b^{x+2}}{(m_cR+m_ar)^x}+\frac{m_c^{x+2}}{(m_aR+m_br)^x}\right)^2=$$

$$=\frac{3}{4}\left(\frac{(m_a^2)^{x+1}}{(m_am_bR+m_cm_ar)^x}+\frac{(m_b^2)^{x+1}}{(m_cm_aR+m_am_br)^x}+\frac{(m_c^2)^{x+1}}{(m_cm_aR+m_bm_cr)^x}\right)^2\ge$$

$$\ge\frac{3}{4}\left(\frac{(m_a^2+m_b^2+mc^2)^{x+1}}{(m_am_bR+m_cm_ar+m_cm_aR+m_am_br+m_cm_aR+m_bm_cr)^x}\right)^2=$$

$$=\frac{3}{4}\cdot\frac{(m_a^2+m_b^2+m_c^2)^{2x+2}}{(R+r)^{2x}(m_am_b+m_bm_c+m_cm_a)^{2x}}\ge\frac{3}{4}\cdot\frac{(m_a^2+m_b^2+m_c^2)^{2x+2}}{(R+r)^{2x}(m_a^2+m_b^2+m_c^2)^{2x}}=$$

$$=\frac{3}{4}\cdot\frac{(m_a^2+m_b^2+m_c^2)^2}{(R+r)^{2x}}=\frac{3}{4(R+r)^{2x}}\cdot\left(\frac{3(a^2+b^2+c^2)}{4}\right)^2\ge$$

$$\ge\frac{3}{4(R+r)^{2x}}\left(\frac{3}{4}\cdot 4\sqrt{3}F\right)^2=\frac{81}{4(R+r)^{2x}}\cdot F^2.$$

Equality holds if and only if ΔABC is equilateral.

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ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.