

ROMANIAN MATHEMATICAL MAGAZINE

S.2419 If $x > 0$, then in $\triangle ABC$ holds:

$$\left(\frac{m_a^{2x+4}}{(m_b R + m_c r)^{2x}} + 2\right) \left(\frac{m_b^{2x+4}}{(m_c R + m_a r)^{2x}} + 2\right) \left(\frac{m_c^{2x+4}}{(m_a R + m_b r)^{2x}} + 2\right) \geq \frac{81}{(R+r)^{2x}} \cdot F^2$$

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Applying Arkady Alt's inequality $(a^2 + t^2)(b^2 + t^2)(c^2 + t^2) \geq \frac{3}{4}t^4(a+b+c)^2$ for $t = \sqrt{2}$,

Radon's inequality, the known inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ and

Ionescu-Weitzenbock's inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$, it follows that

$$\begin{aligned} & \left(\frac{m_a^{2x+4}}{(m_b R + m_c r)^{2x}} + 2\right) \left(\frac{m_b^{2x+4}}{(m_c R + m_a r)^{2x}} + 2\right) \left(\frac{m_c^{2x+4}}{(m_a R + m_b r)^{2x}} + 2\right) \geq \\ & \geq \frac{3}{4}(\sqrt{2})^4 \left(\frac{m_a^{x+2}}{(m_b R + m_c r)^x} + \frac{m_b^{x+2}}{(m_c R + m_a r)^x} + \frac{m_c^{x+2}}{(m_a R + m_b r)^x}\right)^2 = \\ & = 3 \left(\frac{(m_a^2)^{x+1}}{(m_a m_b R + m_c m_a r)^x} + \frac{(m_b^2)^{x+1}}{(m_c m_a R + m_a m_b r)^x} + \frac{(m_c^2)^{x+1}}{(m_c m_a R + m_b m_c r)^x}\right)^2 \geq \\ & \geq 3 \left(\frac{(m_a^2 + m_b^2 + m_c^2)^{x+1}}{(m_a m_b R + m_c m_a r + m_c m_a R + m_a m_b r + m_c m_a R + m_b m_c r)^x}\right)^2 = \\ & = 3 \cdot \frac{(m_a^2 + m_b^2 + m_c^2)^{2x+2}}{(R+r)^{2x}(m_a m_b + m_b m_c + m_c m_a)^{2x}} \geq 3 \cdot \frac{(m_a^2 + m_b^2 + m_c^2)^{2x+2}}{(R+r)^{2x}(m_a^2 + m_b^2 + m_c^2)^{2x}} = \\ & = 3 \cdot \frac{(m_a^2 + m_b^2 + m_c^2)^2}{(R+r)^{2x}} = \frac{3}{(R+r)^{2x}} \cdot \left(\frac{3(a^2 + b^2 + c^2)}{4}\right)^2 \geq \\ & \geq \frac{3}{(R+r)^{2x}} \left(\frac{3}{4} \cdot 4\sqrt{3}F\right)^2 = \frac{81}{(R+r)^{2x}} \cdot F^2. \end{aligned}$$

Equality holds if and only if $\triangle ABC$ is equilateral.

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ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3}{4}t^4(x + y + z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2 + t^2)(y^2 + t^2) \geq \frac{3}{4}t^2((x + y)^2 + t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x - y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$\begin{aligned}(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) &\geq \frac{3t^2}{4}((x + y)^2 + t^2)(t^2 + z^2) \geq \\ &\geq \frac{3t^2}{4}(t(x + y) + tz)^2 = \frac{3}{4}t^4(x + y + z)^2.\end{aligned}$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.