ROMANIAN MATHEMATICAL MAGAZINE

S.2430 If $a,b,c>0, a+b+c=1$, then:

$$\frac{a(b+c)^2}{a+1}+\frac{b(c+a)^2}{b+1}+\frac{c(a+b)^2}{c+1}\leq\frac{1}{3}$$

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Using the tangent line method, we have:

$$\frac{x(1-x)^2}{x+1}\leq\frac{-3x+5}{36}\Leftrightarrow(3x-1)^2(4x-5)\leq 0$$

$$\frac{x(1-x)^2}{x+1}\leq\frac{-3x+5}{36}\quad(1)$$

which is true for $x\leq 1<\frac{5}{4}$. It results that

$$\sum_{cyc}\frac{a(b+c)^2}{a+1}=\sum_{cyc}\frac{a(1-a)^2}{a+1}\overset{(1)}{\leq}\sum_{cyc}\frac{-3a+5}{36}=$$

$$=\frac{-3(a+b+c)+15}{36}=\frac{-3+15}{36}=\frac{1}{3}$$

Equality holds for $a=b=c=\frac{1}{3}$.