S.2436 If $a, b, c > 0$ and $a + b + c = 3$ then find:

$$\max\left\{\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\right\}.$$

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Solution 1. We have

$$\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}-1=$$

$$=\frac{3a}{5a+2b+2c}-\frac{1}{3}+\frac{3b}{2a+5b+2c}-\frac{1}{3}+\frac{3c}{2a+2b+5c}-\frac{1}{3}=$$

$$=\frac{2(a-b)+2(a-c)}{3(5a+2b+2c)}+\frac{2(b-c)+2(b-a)}{3(2a+5b+2c)}+\frac{2(c-a)+2(c-b)}{3(2a+2b+5c)}=$$

$$=\frac{2(a-b)}{3(5a+2b+2c)}-\frac{2(a-b)}{3(2a+5b+2c)}+\frac{2(b-c)}{3(2a+5b+2c)}-\frac{2(b-c)}{3(2a+2b+5c)}$$
$$+\frac{2(c-a)}{3(2a+2b+5c)}-\frac{2(c-a)}{3(5a+2b+2c)}=$$

$$=\frac{2(a-b)(2a+5b+2c-5a-2b-2c)}{3(5a+2b+2c)(2a+5b+2a)}+\frac{2(b-c)(2a+2b+5c-2a-5b-2c)}{3(2a+5b+2c)(2a+2b+5c)}$$
$$+\frac{2(c-a)(5a+2b+2c-5a-2b-2c)}{3(2a+2c+5c)(5a+2b+2c)}=$$

$$=-\frac{2(a-b)^2}{(5a+2b+2c)(2a+5b+2a)}-\frac{2(b-c)^2}{(2a+5b+2c)(2a+2b+5c)}$$
$$-\frac{2(c-a)^2}{(2a+2c+5c)(5a+2b+2c)}\le 0.$$

It results that

$$\max\left\{\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\right\}=1,$$

obtained for $a = b = c = 1$.

Solution 2. We will prove that

$$\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\le 1 \Leftrightarrow \frac{a}{5a+2b+2c}+\frac{b}{2a+5b+2c}+\frac{c}{2a+2b+5c}\le\frac{1}{3}$$

$$\Leftrightarrow \frac{1}{5}-\frac{a}{5a+2b+2c}+\frac{1}{5}-\frac{3b}{2a+5b+2c}+\frac{1}{5}-\frac{3c}{2a+2b+5c}\ge\frac{3}{5}-\frac{1}{3}$$

$$\Leftrightarrow \frac{b+c}{5a+2b+2c}+\frac{c+a}{2a+5b+2c}+\frac{a+b}{2a+2b+5c}\ge\frac{2}{3}.$$

Applying Bergström inequality we obtain

$$\frac{b+c}{5a+2b+2c}+\frac{c+a}{2a+5b+2c}+\frac{a+b}{2a+2b+5c}=$$

$$=\frac{(b+c)^2}{(b+c)(5a+2b+2c)}+\frac{(c+a)^2}{(c+a)(2a+5b+2c)}+\frac{(a+b)^2}{(a+b)(2a+2b+5c)}\ge$$

$$\ge\frac{4(a+b+c)^2}{4(a^2+b^2+c^2)+14(ab+bc+ca)}.$$

It remains to prove that

$$6(a+b+c)^2\ge 4(a^2+b^2+c^2)+14(ab+bc+ca)\Leftrightarrow$$

$$a^2+b^2+c^2\ge ab+bc+ca.$$