ROMANIAN MATHEMATICAL MAGAZINE

S.2444 O – the circumcenter of ΔABC lies on the incircle of ΔABC.

Prove that:

$$8\sqrt{2}+\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}>12$$

Proposed by Daniel Sitaru, Claudia Nănuți – Romania

Solution 1 by Titu Zvonaru-Romania:

Since $OI^2=R^2-2Rr$, we have $R^2-2Rr=r^2 \quad (1)$

By the law of sines and formula $ab+bc+ca=s^2+r^2+4Rr$, we obtain

$$\sin^2 A+\sin^2 B+\sin^2 C=\frac{a^2+b^2+c^2}{4R^2}=\frac{4s^2-2(s^2+r^2+4Rr)}{4R^2}=\frac{s^2-r^2-4Rr}{2R^2} \quad (2)$$

$$\sin A\sin B+\sin B\sin C+\sin C\sin A=\frac{ab+bc+ca}{4R^2}=\frac{s^2+r^2+4Rr}{4R^2} \quad (3)$$

By (2) yields that

$$\cos^2 A+\cos^2 B+\cos^2 C=3-(\sin^2 A+\sin^2 B+\sin^2 C)$$
$$=\frac{6R^2-s^2+r^2+4Rr}{2R^2} \quad (4)$$

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1+\frac{r}{R}=\frac{R+r}{R},$$

it results that

$$2(\cos A\cos B+\cos B\cos C+\cos C\cos A)$$
$$=(\cos A+\cos B+\cos C)^2-(\cos^2 A+\cos^2 B+\cos^2 C)$$
$$=\left(\frac{R+r}{R}\right)^2-\frac{6R^2-s^2+r^2+4Rr}{2R^2}=\frac{s^2-4R^2+r^2}{2R^2} \quad (5)$$

Applying (3) and (5), it follows that

$$\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}=\frac{1}{2}\cos\frac{A-B}{2}\left(\cos\frac{B-C+C-A}{2}+\cos\frac{B-C-C+A}{2}\right)$$
$$=$$
$$=\frac{1}{2}\cos^2\frac{A-B}{2}+\frac{1}{2}\left(\cos\frac{A-B}{2}\cos\frac{B+A-2C}{2}\right)$$
$$=\frac{1}{4}(1+\cos(A-B)+\cos(A-C)+\cos(B-C))$$

$$=\frac{1}{4}\Big(1+(\cos A\cos B+\cos B\cos C+\cos C\cos A)$$
$$+(\sin A\sin B+\sin B\sin C+\sin C\sin A)\Big)=$$

$$=\frac{1}{4}\left(1+\frac{s^2-4R^2+r^2}{4R^2}+\frac{s^2+r^2+4Rr}{4R^2}\right)=\frac{s^2+2Rr+r^2}{8R^2}\quad (6)$$

Using Gerretsen inequality $s^2 \geq 16Rr-5r^2$, Euler inequality $R \geq 2r$ and the relation (1), we get

$$s^2+2Rr+r^2 \geq 16Rr-5r^2+2Rr+r^2=18Rr-4r^2=9(R^2-r^2)-4r^2$$
$$=\frac{23}{4}R^2+\frac{13}{4}(R^2-4r^2) \geq \frac{23}{4}R^2,$$

hence $\frac{s^2+2Rr+r^2}{8R^2} \geq \frac{23}{32}$. It suffices to prove that

$$8\sqrt{2}+\frac{23}{32}>12 \Leftrightarrow 8\sqrt{2}>\frac{361}{32} \Leftrightarrow 131072>130321.$$

Solution 2 by Titu Zvonaru-Romania

Since $OI^2=R^2-2Rr$, we have $R^2-2Rr=r^2\quad (1)$

Using the law of sines we obtain

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \Rightarrow \cos\frac{A-B}{2}=\frac{a+b}{4R\cos\frac{A}{2}}.$$

Since $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sqrt{\frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ca}\cdot\frac{s(s-c)}{ab}}=\frac{sF}{abc}=\frac{sF}{4FR}=\frac{s}{4R}$ and $ab+bc+ca=s^2+r^2+4Rr$, it rezults that

$$\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}=\frac{(a+b)(b+c)(c+a)}{64R^3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$=\frac{(a+b+c)(ab+bc+ca)-abc}{64R^3\cdot\frac{s}{4R}}=$$
$$=\frac{2s(s^2+r^2+4Rr)-4sRr}{16R^2s}=\frac{s^2+r^2+2Rr}{8R^2}.$$

Using Gerretsen inequality $s^2 \geq 16Rr-5r^2$, Euler inequality $R \geq 2r$ and the relation (1), we get

$$s^2+2Rr+r^2 \geq 16Rr-5r^2+2Rr+r^2=18Rr-4r^2=9(R^2-r^2)-4r^2$$
$$=\frac{23}{4}R^2+\frac{13}{4}(R^2-4r^2) \geq \frac{23}{4}R^2,$$

Hence $\frac{s^2+2Rr+r^2}{8R^2} \geq \frac{23}{32}$. It suffices to prove that

$$8\sqrt{2}+\frac{23}{32}>12 \Leftrightarrow 8\sqrt{2}>\frac{361}{32} \Leftrightarrow 131072>130321.$$