ROMANIAN MATHEMATICAL MAGAZINE

S.2454 If $a, b, c > 0, a + b + c = m$ and $ab + bc + ca = n$, prove that

$$\max(a, b, c) - \min(a, b, c) \leq \frac{4}{3}\sqrt{m^2 - 3n}$$

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Suppose that $a \geq b \geq c$; yields that there are $x, y \geq 0$ such that $a = c + x + y, b = c + x$.

We have to prove that:

$$9(x + y)^2 \leq 16((3c + 2x + y)^2 - 3((c + x + y)(c + x) + (c + x)c + c(c + x + y))$$

$$\Leftrightarrow 7x^2 - 2xy + 7y^2 \geq 0 \Leftrightarrow 6(x^2 + y^2) + (x - y)^2 \geq 0.$$

Equality holds if and only if $x = y = 0$, that is $a = b = c, m^2 = 3n$.