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S.2455 In acute-angled ΔABC the following relationship holds:

$$\frac{a}{b+c}\sqrt{tanA}+\frac{b}{c+a}\sqrt{tanB}+\frac{c}{a+b}\sqrt{tanC}\geq\frac{3\sqrt[4]{3}}{2}$$

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Suppose that $a\geq b\geq c$; then $\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}$ and $\tan A\geq\tan B\geq\tan C$.

Applying Cebyshev inequality, Nesbitt's inequality:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

$AM-GM$ inequality and $\tan A\cdot\tan B\cdot\tan C\geq 3\sqrt{3}$ (item 2.32 from $[1]$), we obtain:

$$\frac{a}{b+c}\sqrt{tanA}+\frac{b}{c+a}\sqrt{tanB}+\frac{c}{a+b}\sqrt{tanC}\overset{CEBYSHEV}{\geq}$$

$$\geq\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\sqrt{tanA}+\sqrt{tanB}+\sqrt{tanC}\right)\overset{NESBITT}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}\left(\sqrt{tanA}+\sqrt{tanB}+\sqrt{tanC}\right)\overset{AM-GM}{\geq}\frac{1}{2}\cdot 3\sqrt[3]{\sqrt{tanA}\cdot\sqrt{tanB}\cdot\sqrt{tanC}}=$$

$$=\frac{3}{2}\cdot\sqrt[6]{tanAtanBtanC}\geq\frac{3}{2}\cdot\sqrt[6]{3\sqrt{3}}=\frac{3}{2}\cdot\sqrt[6]{\left(\sqrt{3}\right)^{3}}=\frac{3\sqrt[4]{3}}{2}$$

Equality holds for: $a=b=c$.

[1] O. Bottema, *Geometric Inequalities*, Groningen 1969