S.2456 If $m, n \in R_{+}^{*}$, then in any triangle ABC holds

$$\frac{\cot\frac{A}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\cot\frac{B}{2}}{m+n\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\cot\frac{C}{2}}{m+n\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{9s}{4mR+(m+3n)r}$$

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Using formulas:

$$\tan\frac{A}{2}=\frac{r}{s-a}, ab+bc+ca=s^2+r^2+4Rr, F^2=s(s-a)(s-b)(s-c)$$

we obtain

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{r}{s-a}+\frac{r}{s-b}+\frac{r}{s-c}=$$

$$=\frac{r\big((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{r(3s^2-s(a+b+b+c+c+a)+ab+bc+ca)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{sr(3s^2-4s^2+s^2+r^2+4Rr)}{s(s-a)(s-b)(s-c)}=\frac{Fr(4R+r)}{F^2}=\frac{r(4R+r)}{sr}=\frac{4R+r}{s}$$

and

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r^3}{(s-a)(s-b)(s-c)}=\frac{r^3s}{s(s-a)(s-b)(s-c)}=\frac{Fr^2}{F^2}=\frac{r^2}{sr}=\frac{r}{s}.$$

Applying Bergström inequality it follows that

$$\frac{\cot\frac{A}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\cot\frac{B}{2}}{m+n\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\cot\frac{C}{2}}{m+n\tan\frac{A}{2}\tan\frac{B}{2}}=$$

$$=\frac{\tan\frac{A}{2}\cot\frac{A}{2}}{m\tan\frac{A}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}\cot\frac{B}{2}}{m\tan\frac{B}{2}+n\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}\cot\frac{C}{2}}{m\tan\frac{C}{2}+n\tan\frac{C}{2}\tan\frac{A}{2}\tan\frac{B}{2}}=$$

$$=\frac{1}{m\tan\frac{A}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{1}{m\tan\frac{B}{2}+n\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{1}{m\tan\frac{C}{2}+n\tan\frac{C}{2}\tan\frac{A}{2}\tan\frac{B}{2}}\geq$$

$$\geq\frac{9}{m\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)+3n\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$$=\frac{9}{\frac{m(4R+r)}{s}+\frac{3nr}{s}}=\frac{9s}{4mR+(m+3n)r}.$$

Equality holds if and only if ΔABC is equilateral.