

# ROMANIAN MATHEMATICAL MAGAZINE

**S.2456 If  $m, n \in R_+^*$ , then in any triangle  $ABC$  holds**

$$\frac{\cot \frac{A}{2}}{m + ntan \frac{B}{2} tan \frac{C}{2}} + \frac{\cot \frac{B}{2}}{m + ntan \frac{C}{2} tan \frac{A}{2}} + \frac{\cot \frac{C}{2}}{m + ntan \frac{A}{2} tan \frac{B}{2}} \geq \frac{9s}{4mR + (m + 3n)r}$$

*Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu- Romania*

**Solution by Titu Zvonaru-Romania**

Using formulas:

$$\tan \frac{A}{2} = \frac{r}{s-a}, ab + bc + ca = s^2 + r^2 + 4Rr, F^2 = s(s-a)(s-b)(s-c)$$

we obtain

$$\begin{aligned} \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} &= \frac{r}{s-a} + \frac{r}{s-b} + \frac{r}{s-c} = \\ &= \frac{r((s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a))}{(s-a)(s-b)(s-c)} = \\ &= \frac{r(3s^2 - s(a+b+c+a) + ab + bc + ca)}{(s-a)(s-b)(s-c)} = \end{aligned}$$

$$= \frac{sr(3s^2 - 4s^2 + s^2 + r^2 + 4Rr)}{s(s-a)(s-b)(s-c)} = \frac{Fr(4R+r)}{F^2} = \frac{r(4R+r)}{sr} = \frac{4R+r}{s}$$

and

$$\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} = \frac{r^3}{(s-a)(s-b)(s-c)} = \frac{r^3 s}{s(s-a)(s-b)(s-c)} = \frac{Fr^2}{F^2} = \frac{r^2}{sr} = \frac{r}{s}.$$

Applying Bergström inequality it follows that

$$\begin{aligned} \frac{\cot \frac{A}{2}}{m + ntan \frac{B}{2} tan \frac{C}{2}} + \frac{\cot \frac{B}{2}}{m + ntan \frac{C}{2} tan \frac{A}{2}} + \frac{\cot \frac{C}{2}}{m + ntan \frac{A}{2} tan \frac{B}{2}} &= \\ &= \frac{\tan \frac{A}{2} \cot \frac{A}{2}}{mtan \frac{A}{2} + ntan \frac{A}{2} tan \frac{B}{2} tan \frac{C}{2}} + \frac{\tan \frac{B}{2} \cot \frac{B}{2}}{mtan \frac{B}{2} + ntan \frac{B}{2} tan \frac{C}{2} tan \frac{A}{2}} + \frac{\tan \frac{C}{2} \cot \frac{C}{2}}{mtan \frac{C}{2} + ntan \frac{C}{2} tan \frac{A}{2} tan \frac{B}{2}} = \end{aligned}$$

# ROMANIAN MATHEMATICAL MAGAZINE

$$\begin{aligned} &= \frac{1}{m \tan \frac{A}{2} + n \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} + \frac{1}{m \tan \frac{B}{2} + n \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{A}{2}} + \frac{1}{m \tan \frac{C}{2} + n \tan \frac{C}{2} \tan \frac{A}{2} \tan \frac{B}{2}} \geq \\ &\geq \frac{9}{m \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right) + 3n \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} = \\ &= \frac{9}{\frac{m(4R+r)}{s} + \frac{3nr}{s}} = \frac{9s}{4mR + (m+3n)r}. \end{aligned}$$

Equality holds if and only if  $\triangle ABC$  is equilateral.