ROMANIAN MATHEMATICAL MAGAZINE

S.2457 In acute triangle ABC holds:

$$\sum_{cyc}\frac{\tan A}{bc}\geq 3\sum_{cyc}\frac{\cot A}{bc}$$

Proposed by Marin Chirciu – Romania

Solution by Titu Zvonaru-Romania

By the law of sines, the given inequality is equivalent to

$$\sum_{cyc}\frac{\sin^2 A}{\cos A\sin B\sin C}\geq 3\sum_{cyc}\frac{\cos A}{\sin A\sin B\sin C}\Leftrightarrow$$

$$\sum_{cyc}\sin^2 A\cos B\cos C\geq 3\sum_{cyc}\cos^2 A\cos B\cos C$$

$$\sum_{cyc}(1-\cos^2 A)\cos B\cos C\geq 3\sum_{cyc}\cos^2 A\cos B\cos C\Leftrightarrow$$

$$\sum_{cyc}\cos B\cos C\geq 4\cos A\cos B\cos C\sum_{cyc}\cos A.$$

Using the known identities

$$\sum_{cyc}\cos A\cos B=\frac{r^2+s^2-4R^2}{4R^2},\ \sum_{cyc}\cos A=\frac{R+r}{R},\ \cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2},$$

we have to prove that

$$\frac{r^2+s^2-4R^2}{4R^2}\geq\frac{4(R+r)}{R}\cdot\frac{s^2-(2R+r)^2}{4R^2}$$

$$R(r^2+s^2-4R^2)\geq(4R+4r)(s^2-4R^2-4Rr-r^2)$$

$$12R^3+32R^2r+21Rr^2+4r^3\geq s^2(3R+4).$$

Using Gerretsen inequality $s^2\leq 4R^2+4Rr+3r^2$, it suffices to prove that

$$12R^3+32R^2r+21Rr^2+4r^3\geq(4R^2+4Rr+3r^2)(3R+4r)$$

$$r(R^2-Rr-2r^2)\geq 0\Leftrightarrow r(R-2r)(R+r)\geq 0,$$

which is true because $R\geq 2r$ (Euler inequality).

Equality holds if and only if the triangle ABC is equilateral.