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S.2466 In ΔABC the following relationship holds:

$$h_a^2 \cdot \cot\frac{A}{2} + h_b^2 \cdot \cot\frac{B}{2} + h_c^2 \cdot \cot\frac{C}{2} \ge 9F$$

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Since $ah_a = 2F, \cot\frac{A}{2} = \frac{s-a}{r}, F = sr$, the inequality is equivalent to

$$\frac{4F^2}{a^2} \cdot \frac{s-a}{r} + \frac{4F^2}{b^2} \cdot \frac{s-b}{r} + \frac{4F^2}{c^2} \cdot \frac{s-c}{r} \ge 9F \Leftrightarrow \frac{s-a}{a^2 r} + \frac{s-b}{b^2 r} + \frac{s-c}{c^2 r} \ge \frac{9}{4F} \Leftrightarrow$$

$$\frac{s-a}{a^2} + \frac{s-b}{b^2} + \frac{s-c}{c^2} \ge \frac{9}{4s} \qquad (1)$$

Here three proofs for (1):

I. Applying Bergström inequality we obtain

$$\frac{s-a}{a^2} + \frac{s-b}{b^2} + \frac{s-c}{c^2} = \frac{(s-a)^2}{a^2(s-a)} + \frac{(s-b)^2}{b^2(s-b)} + \frac{(s-c)^2}{c^2(s-c)}$$
$$\ge \frac{(s-a+s-b+s-c)^2}{a^2(s-a) + b^2(s-b) + c^2(s-c)}.$$

It remains to prove that

$$\frac{s^2}{a^2(s-a) + b^2(s-b) + c^2(s-c)} \ge \frac{9}{4s}$$

$$4s^3 \ge 9s(a^2 + b^2 + c^2) - 9(a^3 + b^3 + c^3)$$

$$(a+b+c)^3 + 18(a^3 + b^3 + c^3) \ge 9(a+b+c)(a^2 + b^2 + c^2)$$

$$5(a^3 + b^3 + c^3) + 3abc \ge 3(a^2 b + ab^2 + b^2 c + bc^2 + a^2 c + ac^2) \qquad (2)$$

By Schur inequality $a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \ge 0$ it follows that $a^3 + b^3 + c^3 + 3abc \ge a^2 b + ab^2 + b^2 c + bc^2 + a^2 c + ac^2$, and by Muirhead inequality we have

$$4\sum_{\text{sym}} a^3 \ge 4\sum_{\text{sym}} a^2 b.$$

It results that the inequality (2) is true. Equality holds if and only if $a = b = c$, that is if and only if ΔABC is equilateral.

II. **We have**

$$\sum_{cyc}\frac{s-a}{a^2}-\frac{9}{4s}=\frac{1}{2}\sum_{cyc}\left(\frac{b+c-a}{a^2}-\frac{3}{a+b+c}\right)=\frac{1}{2}\sum_{cyc}\frac{(b+c-a)(b+c+a)-3a^2}{a^2(a+b+c)}=$$

$$=\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(b+c)^2-4a^2}{a^2}=\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(b+c+2a)(b-a+c-a)}{a^2}=$$

$$=\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(b+c+2a)(b-a)}{a^2}+\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(b+c+2a)(c-a)}{a^2}=$$

$$=\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(b+c+2a)(b-a)}{a^2}+\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(c+a+2b)(a-b)}{b^2}=$$

$$=\frac{1}{2(a+b+c)}\sum_{cyc}(a-b)\left(\frac{c+a+2b}{b^2}-\frac{b+c+2a}{a^2}\right)$$

$$=\frac{1}{2(a+b+c)}\sum_{cyc}\frac{(a-b)^2(a^2+b^2+3ab+bc+ac)}{a^2b^2}\geq 0.$$

Equality if and only if $a=b=c$.

III. **Using the inequality $x^2+y^2+z^2\geq xy+yz+zx$ applied for $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$, Gerretsen inequality $s^2\geq 16Rr-5r^2$, Euler inequality $R\geq 2r$ and formula $ab+bc+ca=s^2+r^2+4Rr$, it follows that**

$$\begin{aligned}\frac{s-a}{a^2}+\frac{s-b}{b^2}+\frac{s-c}{c^2}&=s\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\\&\geq s\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{2s^2-s^2-r^2-4Rr}{4sRr}\\&=\frac{s^2-r^2-4Rr}{4sRr}\geq\frac{16Rr-5r^2-r^2-4Rr}{4sRr}=\frac{12R-6r}{4sR}\\&=\frac{9R+3(R-2r)}{4sR}\geq\frac{9R}{4sR}=\frac{9}{4s}.\end{aligned}$$

Equality if and only if ΔABC is equilateral.