ROMANIAN MATHEMATICAL MAGAZINE

S.2470 If $a,b,c>0$ and $a+b+c=3$ then find

$$\max\left\{\sum_{cyc}\frac{a}{a^2+3}+\sum_{cyc}\frac{a^2}{a^3+1}\right\}$$

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We will prove that

$$\sum_{cyc}\left(\frac{a}{a^2+3}+\frac{a^2}{a^3+1}\right)\le\frac{9}{4}\quad(1)$$

By tangent line method, we search m,n such that

$$\frac{x}{x^2+3}+\frac{x^2}{x^3+1}\le mx+n\quad(2)$$

For $x=1$, from (2) we get $m+n=\frac{3}{4}$ (3)

By derivation, the inequality (2) becomes

$$\frac{x^2+3-2x^2}{(x^2+3)^2}+\frac{2x(x^3+1)-3x^4}{(x^3+1)^2}\le m\quad(4)$$

For $x=1$, from (4) we obtain $m=\frac{3}{8}$. Now, by (3) yields $n=\frac{3}{8}$.

For $m=n=\frac{3}{8}$, the inequality (2) is equivalent to

$$\frac{x}{x^2+3}+\frac{x^2}{x^3+1}\le\frac{3(x+1)}{8}\quad(5)$$

$$3x^6+3x^5-7x^4+12x^3-21x^2+x+9\ge0$$

$$(x-1)^2(3x^4+9x^3+8x^2+19x+9)\ge0,$$

which is true for $x>0$.

Writting the inequality (5) for a,b,c, and adding up, it follows that

$$\sum_{cyc}\left(\frac{a}{a^2+3}+\frac{a^2}{a^3+1}\right)\le\frac{3(a+1)}{8}+\frac{3(b+1)}{8}+\frac{3(c+1)}{8}=\frac{3(a+b+c)}{8}+\frac{9}{8}=\frac{9}{4}.$$

The searched maximum is equal to $\frac{9}{4}$, obtained for $a=b=c=1$.