ROMANIAN MATHEMATICAL MAGAZINE

S.2472 If $a, b, c > 0$, then:

$$(2a+b)\sqrt{\frac{a}{b}}+(2b+c)\sqrt{\frac{b}{c}}+(2c+a)\sqrt{\frac{c}{a}}\geq 3(a+b+c)$$

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First, we will prove the inequality:

$$\frac{x^3}{y}+\frac{y^3}{z}+\frac{z^3}{x}\geq x^2+y^2+z^2 \quad (1)$$

Using the known inequality $x^2+y^2+z^2\geq xy+yz+zx$, we obtain

$$\frac{x^3}{y}+\frac{y^3}{z}+\frac{z^3}{x}=$$

$$=\frac{x^3}{y}-2x^2+xy+\frac{y^3}{z}-2y^2+yz+\frac{z^3}{x}-2z^2+zx+2(x^2+y^2+z^2)-(xy+yz+zx)$$

$$=\frac{x(x-y)^2}{y}+\frac{y(y-z)^2}{z}+\frac{z(z-x)^2}{x}+\left(x^2+y^2+z^2-xy-yz-zx\right)+x^2+y^2+z^2\geq$$

$$\geq x^2+y^2+z^2,$$

hence the inequality (1) is true. Equality holds if and only if $x=y=z$.

Applying $AM-GM$ inequality and (1) for $x=\sqrt{a}, y=\sqrt{b}, z=\sqrt{c}$, it follows that

$$(2a+b)\sqrt{\frac{a}{b}}+(2b+c)\sqrt{\frac{b}{c}}+(2c+a)\sqrt{\frac{c}{a}}=$$

$$=(a+a+b)\sqrt{\frac{a}{b}}+(b+b+c)\sqrt{\frac{b}{c}}+(c+c+a)\sqrt{\frac{c}{a}}\geq$$

$$\geq\left(a+2\sqrt{ab}\right)\sqrt{\frac{a}{b}}+\left(b+2\sqrt{bc}\right)\sqrt{\frac{b}{c}}+\left(c+2\sqrt{ca}\right)\sqrt{\frac{c}{a}}=$$

$$=\frac{\left(\sqrt{a}\right)^3}{\sqrt{b}}+\frac{\left(\sqrt{b}\right)^3}{\sqrt{c}}+\frac{\left(\sqrt{c}\right)^3}{\sqrt{a}}+2a+2b+2c\geq a+b+c+2(a+b+c)=3(a+b+c).$$

Equality holds if and only if $a=b=c$.