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S.2478 If $a, b, c > 0$ and $n \in N$ then:

$$\sum_{cyc} \frac{1}{a^n b^n (a+b)} \ge \frac{9}{2(a^{n+1} + b^{n+1} + c^{n+1})}$$

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We have:

$$a^n b^n (a+b) \le a^{n+1} + b^{n+1} \Leftrightarrow a^n (a-b) + b^n (b-a) \ge 0 \Leftrightarrow (a-b)(a^n - b^n) \ge 0$$
$$\Leftrightarrow (a-b)^2 (a^{n-1} + a^{n-2} b + \cdots + a b^{n-2} + b^{n-1}) \ge 0,$$

which is obviously true. Yields that

$$a^n b^n (a+b) \le a^{n+1} + b^{n+1} \quad (1)$$

Applying (1) and Bergström inequality, it follows that

$$\sum_{cyc} \frac{1}{a^n b^n (a+b)} \ge \sum_{cyc} \frac{1}{a^{n+1} + b^{n+1}} \ge \frac{9}{2(a^{n+1} + b^{n+1} + c^{n+1})}.$$

Equality holds if and only if $a = b = c$.