ROMANIAN MATHEMATICAL MAGAZINE

S.2479 Let $a,b,c\geq 0$ such that $ab+bc+ca=1$. Prove that

$$\frac{4(a^3+b^3+c^3)}{3}\geq a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)\geq\frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

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Solution by Titu Zvonaru-Romania

$$a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)=$$

$$=a-ab^2-ac^2+ab^2c^2+b-bc^2-a^2b+a^2bc^2+c-a^2c-b^2c+a^2b^2c=$$

$$=(a+b+c)(ab+bc+ca)+abc(ab+bc+ca)-a^2b-ab^2-b^2c-bc^2-c^2a-ca^2=4abc$$

and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{1}{abc}.$$

The inequality is equivalent to:

$$\frac{a^3+b^3+c^3}{3}\geq abc\geq 27a^3b^3c^3 \qquad (1)$$

The left inequality from (1) is $AM-GM$ inequality.

Equality holds if and only if $a=b=c=\frac{1}{\sqrt{3}}$.

Applying again $AM-GM$ inequality, it follows that

$$1=ab+bc+ca\geq 3(a^2b^2c^2)^{1/3}$$

that is $1\geq 27a^2b^2c^2$; the right inequality from (1) is proved.

Equality holds if and only if $a=b=c=\frac{1}{\sqrt{3}}$.