ROMANIAN MATHEMATICAL MAGAZINE

S.2480 If $a,b,c>0$ then:

$$\frac{1}{\sum_{\text{cyc}} a^2b+3abc}+\frac{1}{\sum_{\text{cyc}} ab^2+3abc}\geq\frac{27}{4(a+b+c)^3-27abc}$$

Proposed by Sidi Abdallah Lemrabott - Mauritania

Solution by Titu Zvonaru-Romania

By Bergström's inequality we have:

$$\frac{1}{\sum_{\text{cyc}} a^2b+3abc}+\frac{1}{\sum_{\text{cyc}} ab^2+3abc}\geq\frac{4}{\sum_{\text{cyc}} a^2b+\sum_{\text{cyc}} ab^2+6abc}$$

It suffices to prove that

$$\frac{4}{\sum_{\text{cyc}} a^2b+\sum_{\text{cyc}} ab^2+6abc}\geq\frac{27}{4(a+b+c)^3-27abc}$$

which is equivalent to

$$16(a^3+b^3+c^3)+48(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+96abc-108abc$$
$$\geq 27(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+162abc$$

$$16(a^3+b^3+c^3)+21(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)\geq 174abc \quad (1)$$

By $AM-GM$ inequality yields that

$$16(a^3+b^3+c^3)\geq 48abc$$

$$21(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)\geq 126abc$$

hence the inequality (1) is true.

Equality holds if and only if $a=b=c$.