ROMANIAN MATHEMATICAL MAGAZINE

S.2482 In triangle ABC, ω - Brocard's angle. Prove that

$$\csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \geq 13.$$

When does equality holds ?

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It is known that $\csc^2\omega = \frac{a^2b^2+b^2c^2+c^2a^2}{4F^2}$.

Using the sines law and the formula $R = \frac{abc}{4F}$, the given inequality is equivalent to

$$\frac{a^2b^2+b^2c^2+c^2a^2}{4F^2} + \frac{4(a^2+b^2+c^2)}{4R^2} \geq 13$$

$$\frac{a^2b^2+b^2c^2+c^2a^2}{4F^2} + \frac{64F^2(a^2+b^2+c^2)}{4a^2b^2c^2} \geq 13$$

$$a^2b^2c^2(a^2b^2+b^2c^2+c^2a^2) + 64F^4(a^2+b^2+c^2) \geq 52a^2b^2c^2F^2 \quad (1)$$

Applying the Heron formula $16F^2 = 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$, the inequality (1) becomes

$$4a^2b^2c^2(a^2b^2+b^2c^2+c^2a^2) +$$
$$+(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)^2(a^2+b^2+c^2) \geq$$
$$\geq 13a^2b^2c^2(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4) \quad (2)$$

Denoting $x = a^2, y = b^2, z = c^2$, we have to prove that:

$$4xyz(xy+yz+zx) + (2xy+2yz+2zx-x^2-y^2-z^2)^2(x+y+z) \geq$$
$$\geq 13xyz(2xy+2yz+2zx-x^2-y^2-z^2) \quad (3)$$

The inequality (3) is simetric of 5 degree; it suffices to prove it for $z=0$ and for $y=z$.

For $z=0$, the inequality (3) is obviously true. For $y=z$, the inequality (3) is

$$4xy^2(2xy+y^2) + (4xy-x^2)^2(x+2y) \geq 13xy^2(4xy-x^2)$$

$$x(x^4-6x^3y+13x^2y^2-12xy^3+4y^4) \geq 0$$

$$x(x-y)^2(x-2y)^2 \geq 0,$$

which is true.

Equality holds if and only if the triangle ABC is equilateral.