ROMANIAN MATHEMATICAL MAGAZINE

S.2489 If $x, y, z, n \in N, x^2 + y^2 + z^2 \leq 9$ then:

$$2^n \cdot x^{n+1} + 3^n \cdot y^{n+1} + 6^n \cdot z^{n+1} \geq 3^n (xyz)^{\frac{n+3}{3}}$$

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Applying Radon's inequality, the known inequality $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$

and $AM - GM$ inequality, it follows that:

$$2^n \cdot x^{n+1} + 3^n \cdot y^{n+1} + 6^n \cdot z^{n+1} = \frac{x^{n+1}}{\left(\frac{1}{2}\right)^n} + \frac{y^{n+1}}{\left(\frac{1}{3}\right)^n} + \frac{z^{n+1}}{\left(\frac{1}{6}\right)^n} \geq$$

$$\geq \frac{(x + y + z)^{n+1}}{\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)^n} = (x + y + z)^n =$$

$$= \frac{(x + y + z)^{n+3}}{(x + y + z)^2} \geq \frac{(x + y + z)^{n+3}}{3(x^2 + y^2 + z^2)} \geq \frac{(x + y + z)^{n+3}}{3^3} \geq \frac{3^{n+3}(xyz)^{\frac{n+3}{3}}}{3^3} = 3^n (xyz)^{\frac{n+3}{3}}$$

Equality holds if and only if $x = y = z = \sqrt{3}$ and $2^n + 3^n + 6^n = 3^n$, that is $n = 0$.