ROMANIAN MATHEMATICAL MAGAZINE

**S.2507 Let $x, y, z$ be positive real numbers such that $x+y+z=3$ and $m\le 9$. Prove that:**

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{mxyz}{xy+yz+zx}\ge 3+\frac{m}{3}$$

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We have

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-9=\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}-6=$$

$$=\frac{(x-y)^2}{xy}+\frac{(y-z)^2}{yz}+\frac{(z-x)^2}{zx}\quad (1)$$

and

$$(x+y+z)(xy+yz+zx)-9xyz=z(x-y)^2+x(y-z)^2+y(z-x)^2\quad (2)$$

Since $x+y+z=3$, using $(1)$ and $(2)$, the desired inequality is equivalent to

$$(x+y+z)^2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{9mxyz}{xy+yz+zx}\ge (9+m)(x+y+z)$$

$$(x+y+z)\left((x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-9\right)\ge m\left(x+y+z-\frac{9xyz}{xy+yz+zx}\right)$$

$$(x+y+z)\left(\frac{(x-y)^2}{xy}+\frac{(y-z)^2}{yz}+\frac{(z-x)^2}{zx}\right)\ge m\cdot\frac{z(x-y)^2+x(y-z)^2+y(z-x)^2}{xy+yz+zx}.$$

It easy to see that it suffices to prove that

$$(x+y+z)\cdot\frac{(x-y)^2}{xy}\ge m\cdot\frac{z(x-y)^2}{xy+yz+zx}\quad (3)$$

If $x=y$ then it holds equality, else it remains to prove that

$$(x+y+z)(xy+yz+zx)\ge m\cdot xyz\quad (4)$$

Since $9\ge m$, applying $(2)$ it follows that $(4)$ is true.

Equality holds if and only if $x=y=z=1$.