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S.2516 If $m, n \in R_+^*$ then prove that in any triangle ABC holds:

$$\frac{\cot^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{m\tan\frac{C}{2}+n\tan\frac{A}{2}}+\frac{\cot^3\frac{C}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}}\geq\frac{s^2}{(m+n)r^2}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Titu Zvonaru-Romania

Applying Bergstrom's inequality, it follows that:

$$\frac{\cot^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{m\tan\frac{C}{2}+n\tan\frac{A}{2}}+\frac{\cot^3\frac{C}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}}=$$

$$=\frac{\tan\frac{A}{2}\cot^3\frac{A}{2}}{m\tan\frac{A}{2}\tan\frac{B}{2}+n\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\tan\frac{B}{2}\cot^3\frac{B}{2}}{m\tan\frac{B}{2}\tan\frac{C}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}}+\frac{\tan\frac{C}{2}\cot^3\frac{C}{2}}{m\tan\frac{C}{2}\tan\frac{A}{2}+n\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$$=\frac{\cot^2\frac{A}{2}}{m\tan\frac{A}{2}\tan\frac{B}{2}+n\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{\cot^2\frac{B}{2}}{m\tan\frac{B}{2}\tan\frac{C}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}}+\frac{\cot^2\frac{C}{2}}{m\tan\frac{C}{2}\tan\frac{A}{2}+n\tan\frac{B}{2}\tan\frac{C}{2}}\geq$$

$$\geq\frac{(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2})^2}{(m+n)(\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2})}\quad(1)$$

Since $\tan\frac{A}{2}=\frac{r}{s-a}$, $\cot\frac{A}{2}=\frac{s-a}{r}$, we obtain:

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s-a+s-b+s-c}{r}=\frac{s}{r}\quad(2)$$

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=\frac{r^2}{(s-a)(s-b)}+\frac{r^2}{(s-b)(s-c)}+\frac{r^2}{(s-c)(s-a)}=$$

$$=\frac{r^2(s-a+s-b+s-c)}{(s-a)(s-b)(s-c)}=\frac{r^2s^2}{s(s-a)(s-b)(s-c)}=\frac{F^2}{F^2}=1\quad(3)$$

Using (1), (2) and (3) yields

$$\frac{\cot^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{m\tan\frac{C}{2}+n\tan\frac{A}{2}}+\frac{\cot^3\frac{C}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}}\geq\frac{\frac{s^2}{r^2}}{m+n}=\frac{s^2}{(m+n)r^2}.$$

Equality holds if and only if the triangle ABC is equilateral.

Comment: A second proof for (3):

$$\frac{A}{2}+\frac{B}{2}=90^\circ-\frac{C}{2}\Leftrightarrow \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\frac{C}{2}\Leftrightarrow \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}\Leftrightarrow$$

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$