ROMANIAN MATHEMATICAL MAGAZINE

S.2523 If $m \geq 0, x, y, z > 0, xyz = 1$, then in the triangle ABC holds:

$$\frac{x+y}{(h_a h_b)^{m+1}} + \frac{y+z}{(h_b h_c)^{m+1}} + \frac{z+x}{(h_c h_a)^{m+1}} \geq \frac{2\left(\sqrt{3}\right)^{1-m}}{F^{m+1}}$$

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We have $ah_a = bh_b = ch_c = 2F$. Applying $AM - GM$ inequality, Cesaro's inequality $(x+y)(y+z)(z+x) \geq 8xyz = 8$ and Carlitz's inequality $(abc)^{2/3} \geq \frac{4}{\sqrt{3}}F$,

it follows that:

$$\frac{x+y}{(h_a h_b)^{m+1}} + \frac{y+z}{(h_b h_c)^{m+1}} + \frac{z+x}{(h_c h_a)^{m+1}} =$$

$$= \frac{(ab)^{m+1}(x+y)}{(ah_a bh_b)^{m+1}} + \frac{(bc)^{m+1}(y+z)}{(bh_b ch_c)^{m+1}} + \frac{(ca)^{m+1}(z+x)}{(ch_c ah_a)^{m+1}} =$$

$$\geq \frac{3}{(2F)^{2m+2}}\left((x+y)(y+z)(z+x)(abc)^{2m+2}\right)^{\frac{1}{3}} \geq \frac{6}{(2F)^{2m+2}}\left((abc)^{\frac{2}{3}}\right)^{m+1}$$

$$\geq \frac{6}{(2F)^{2m+2}}\left(\frac{4}{\sqrt{3}}F\right)^{m+1} = \frac{2\left(\sqrt{3}\right)^{1-m}}{F^{m+1}}$$

Equality holds if only if the triangle ABC is equilateral.