ROMANIAN MATHEMATICAL MAGAZINE

S.2542 Solve for real numbers

$$x+y+z=-1$$
$$x^3+y^3+z^3=-1$$
$$x^2+2y^2-3yz=2$$

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From the first equation yields $z=-1-x-y$; then the second equation becomes

$$x^3+y^3-(x+y+1)^3=-1$$
$$x^3+y^3-x^3-y^3-1-3x^2y-3xy^2-3x^2-3x-3y^2-3y-6xy=-1$$
$$x^2y+xy^2+x^2+x+y^2+y+xy=0 \Leftrightarrow xy(x+y)+(x+y)^2+(x+y)=0$$
$$\Leftrightarrow (x+y)(xy+x+y+1)=0 \Leftrightarrow (x+y)(x+1)(y+1)=0.$$

We have three cases:

a) $y=-x, z=-1$; the third equation is $x^2+2x^2-3x=2$. It results the solutions $(x,y,z)=\left(\frac{3+\sqrt{33}}{6},-\frac{3+\sqrt{33}}{6},-1\right),\left(\frac{3-\sqrt{33}}{6},-\frac{3-\sqrt{33}}{6},-1\right)$

b) $x=-1, z=-y$; the third equation is $1+2y^2+3y^2=2$. Yields the solutions $(x,y,z)=\left(-1,\frac{\sqrt{5}}{5},-\frac{\sqrt{5}}{5}\right),\left(-1,-\frac{\sqrt{5}}{5},\frac{\sqrt{5}}{5}\right)$

c) $y=-1, z=-x$; the third equation is $x^2+2-3x=2$. We get solutions $(x,y,z)=(0,-1,0),(3,-1,-3)$.