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S.2554 If $m,n,t,x,y,z>0$, then:

$$\left(\left(\frac{x}{my+nz}+\frac{y}{mz+nx}\right)^2+t^2\right)\cdot\left(\left(\frac{y}{mz+nx}+\frac{z}{mx+ny}\right)^2+t^2\right)\cdot$$

$$\cdot\left(\left(\frac{z}{mx+ny}+\frac{x}{my+nz}\right)^2+t^2\right)\geq\frac{27}{(m+n)^2}\cdot t^4$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that:

$$\left(\left(\frac{x}{my+nz}+\frac{y}{mz+nx}\right)^2+t^2\right)\cdot\left(\left(\frac{y}{mz+nx}+\frac{z}{mx+ny}\right)^2+t^2\right)\cdot$$

$$\cdot\left(\left(\frac{z}{mx+ny}+\frac{x}{my+nz}\right)^2+t^2\right)\geq$$

$$\geq\frac{3}{4}\cdot t^4\left(\frac{x}{my+nz}+\frac{y}{mz+nx}+\frac{y}{mz+nx}+\frac{z}{mx+ny}+\frac{z}{mx+ny}+\frac{x}{my+nz}\right)^2\geq$$

$$\geq 3\cdot t^4\left(\frac{x}{my+nz}+\frac{y}{mz+nx}++\frac{z}{mx+ny}\right)^2\quad(1)$$

Using Bergström inequality and the known inequality $(x+y+z)^2\geq 3(xy+yz+zx)$, we obtain

$$\frac{x}{my+nz}+\frac{y}{mz+nx}+\frac{z}{mx+ny}=$$

$$=\frac{x^2}{mxy+nzx}+\frac{y^2}{myz+nxy}+\frac{z^2}{mzx+nyz}\geq\frac{(x+y+z)^2}{(m+n)(xy+yz+zx)}\geq\frac{3}{m+n}\quad(2)$$

By (1) and (2) it results that

$$\left(\left(\frac{x}{my+nz}+\frac{y}{mz+nx}\right)^2+t^2\right)\cdot\left(\left(\frac{y}{mz+nx}+\frac{z}{mx+ny}\right)^2+t^2\right)\cdot$$

$$\cdot\left(\left(\frac{z}{mx+ny}+\frac{x}{my+nz}\right)^2+t^2\right)\geq$$

$$\geq 3t^4\left(\frac{3}{m+n}\right)^2 = \frac{27}{(m+n)^2}\cdot t^4$$

Equality holds if and only if $x = y = z$ and $\frac{1}{m+n} + \frac{1}{m+n} = \frac{t}{\sqrt{2}}$ that is

if and only if $x = y = z, m + n = \frac{2\sqrt{2}}{t}$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.