ROMANIAN MATHEMATICAL MAGAZINE

S.2555 If $x, y, z > 0$, then:

$$\left(\left(\frac{x}{y+z}\right)^2+2\right)\left(\left(\frac{y}{z+x}\right)^2+2\right)\left(\left(\frac{z}{x+y}\right)^2+2\right)\geq 3\left(\frac{x+y}{x+y+2z}+\frac{y+z}{y+z+2x}+\frac{z+x}{z+x+2y}\right)^2\geq\frac{27}{4}$$

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First of all, we will prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}=\frac{(a-b)^2}{2(a+c)(b+c)}+\frac{(b-c)^2}{2(b+a)(c+a)}+\frac{(c-a)^2}{2(c+b)(a+b)}\quad (1)$$

We have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}=\frac{a}{b+c}-\frac{1}{2}+\frac{b}{c+a}-\frac{1}{2}+\frac{c}{a+b}-\frac{1}{2}=$$

$$=\frac{a-b+a-c}{2(b+c)}+\frac{b-c+b-a}{2(c+a)}+\frac{c-a+c-b}{2(a+b)}=$$

$$=\frac{a-b}{2}\left(\frac{1}{b+c}-\frac{1}{c+a}\right)+\frac{b-c}{2}\left(\frac{1}{c+a}-\frac{1}{a+b}\right)+\frac{c}{2}\left(\frac{1}{a+b}-\frac{1}{b+c}\right)=$$

$$=\frac{(a-b)^2}{2(a+c)(b+c)}+\frac{(b-c)^2}{2(b+a)(c+a)}+\frac{(c-a)^2}{2(b+c)(a+b)}.$$

Using the relationship (1) for $(a,b,c)=(x,y,z)$ and for $(a,b,c)=(y+z,z+x,x+y)$, we obtain

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}-\frac{3}{2}=\frac{(x-y)^2}{2(x+z)(y+z)}+\frac{(y-z)^2}{2(y+x)(z+x)}+\frac{(z-x)^2}{2(y+x)(y+z)}\quad (2)$$

$$\frac{x+y}{x+y+2z}+\frac{y+z}{y+z+2x}+\frac{z+x}{z+x+2y}-\frac{3}{2}=$$

$$=\frac{(x-y)^2}{2(z+x+2y)(y+z+2x)}+\frac{(y-z)^2}{2(z+x+2y)(x+y+2z)}+\frac{(z-x)^2}{2(y+z+2x)(x+y+2z)}\quad (3)$$

Applying Arkady Alt's inequality

$$(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2\qquad(4)$$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that

$$\left(\left(\frac{x}{y+z}\right)^2+2\right)\left(\left(\frac{y}{z+x}\right)^2+2\right)\left(\left(\frac{z}{x+y}\right)^2+2\right)$$
$$\geq\frac{3}{4}(\sqrt{2})^2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2\qquad(5)$$

By (5), yields that we have to prove that

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{x+y}{x+y+2z}+\frac{y+z}{y+z+2x}+\frac{z+x}{z+x+2y}\geq\frac{3}{2}\qquad(6)$$

Using (2) and (3), the left inequality from (6) is equivalent to

$$\frac{(x-y)^2}{2(x+z)(y+z)}+\frac{(y-z)^2}{2(y+x)(z+x)}+\frac{(z-x)^2}{2(y+x)(y+z)}\geq$$
$$\geq\frac{(x-y)^2}{2(z+x+2y)(y+z+2x)}+\frac{(y-z)^2}{2(z+x+2y)(x+y+2z)}$$
$$+\frac{(z-x)^2}{2(y+z+2x)(x+y+2z)}\qquad(7)$$

It obviously that it suffices to prove that

$$\frac{(x-y)^2}{(x+z)(y+z)}\geq\frac{(x-y)^2}{(z+x+2y)(y+z+2x)}\qquad(8)$$

If $x=y$ then there is equality, else it remains to prove that

$$(z+x+2y)(y+z+2x)\geq(x+z)(y+z),$$

which is true.

The right inequality from (6) follows easy by (3).

The inequality (5) is strict. We deduce that the left inequality is strict, and for the right inequality, equality holds if and only if $x=y=z$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.