ROMANIAN MATHEMATICAL MAGAZINE

S.2556 If $x,y\geq 0$ and $a,b,c,x+y>0$ then:

$$(a^2x^2+b^2y^2+2)(b^2x^2+c^2y^2+2)(c^2x^2+a^2y^2+2)\geq \frac{9}{2}(x+y)^2(ab+bc+ca)$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2\geq \frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$) it follows that

$$(a^2x^2+b^2y^2+2)(b^2x^2+c^2y^2+2)(c^2x^2+a^2y^2+2)\geq$$

$$\geq \frac{3}{4}\left(\sqrt{2}\right)^4\left(\sqrt{a^2x^2+b^2y^2}+\sqrt{b^2x^2+c^2y^2}+\sqrt{c^2x^2+a^2y^2}\right)^2 \quad (1)$$

By Minkovski inequality, $QM-AM$ inequality $\sqrt{\frac{x^2+y^2}{2}}\geq \frac{x+y}{2}$, and the known inequality $(a+b+c)^2\geq 3(ab+bc+ca)$, we obtain

$$\sqrt{a^2x^2+b^2y^2}+\sqrt{b^2x^2+c^2y^2}+\sqrt{c^2x^2+a^2y^2}\geq$$

$$\geq \sqrt{(ax+bx+cx)^2+(by+cy+ay)^2}=\sqrt{(x^2+y^2)(a+b+c)^2}>$$

$$>\frac{x+y}{\sqrt{2}}\sqrt{3(ab+bc+ca)} \quad (2)$$

Using (1) and (2) it results that

$$(a^2x^2+b^2y^2+2)(b^2x^2+c^2y^2+2)(c^2x^2+a^2y^2+2)\geq$$

$$\geq 3\left(\frac{x+y}{\sqrt{2}}\sqrt{3(ab+bc+ca)}\right)^2=\frac{9}{2}(x+y)^2(ab+bc+ca).$$

Equality holds if and only if $x=y,\sqrt{x^2(a^2+b^2)}=\sqrt{x^2(b^2+c^2)}=\sqrt{x^2(c^2+a^2)}=1$, that is $x=y,a=b=c=\frac{1}{x\sqrt{2}}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.