ROMANIAN MATHEMATICAL MAGAZINE

S.2558 If $t,a,b,c,d>0$ then

$$(a^2+t^2)(b^2+t^2)c^2+t^2)(d^2+t^2)>\frac{9}{16}(a+b+c+d)^2\cdot t^6$$

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We will prove that if $t,x,y>0$, then

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\quad(1)$$

The inequality (1) is equivalent to

$$\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0,$$

obviously true.

Applying (1) and Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$ with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$, it follows that

$$(a^2+t^2)(b^2+t^2)c^2+t^2)(d^2+t^2)\geq\frac{3}{4}t^2((a+b)^2+t^2)(c^2+t^2)(d^2+t^2)\geq$$

$$\geq\frac{3}{4}t^2\cdot\frac{3}{4}t^4(a+b+c+d)^2=\frac{9}{16}(a+b+c+d)^2\cdot t^6$$

Equality holds if and only if $a=b=\frac{t}{\sqrt{2}}$ and $a+b=c=d=\frac{t}{\sqrt{2}}$, contradiction, hence the inequality is strict.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.