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S.2560 If $t,x,y,z>0$ then

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+t\right)\left(\frac{1}{(y+z)^4}+t\right)\left(\frac{1}{(z+x)^4}+t\right)\ge\frac{729}{64}\cdot t^2$$

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Applying twice Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$), it follows that

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+t\right)\left(\frac{1}{(y+z)^4}+t\right)\left(\frac{1}{(z+x)^4}+t\right)$$
$$\ge\frac{3}{4}\left(\sqrt{2}\right)^4(xy+yz+zx)^2\cdot\frac{3}{4}\left(\sqrt{t}\right)^4\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2=$$
$$=\frac{9}{4}\left((xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\right)^2\cdot t^2\quad(1)$$

The inequality

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}\quad(2)$$

is a famous one, known as Iran 1996. In fact, it was proposed by Ji Chen in Crux Mathematicorum nr. 4/1994.

By (1) and (2) it results that

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+t\right)\left(\frac{1}{(y+z)^4}+t\right)\left(\frac{1}{(z+x)^4}+t\right)\ge\frac{729}{64}\cdot t^2$$

Equality holds if and only if $x=y=z, xy=yz=zx=1, \frac{1}{(x+y)^2}=\frac{1}{(y+z)^2}=\frac{1}{(z+x)^2}=\frac{\sqrt{t}}{\sqrt{2}}$, that is $x=y=z=1, t=\frac{1}{8}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.