ROMANIAN MATHEMATICAL MAGAZINE

S.2561 If $x,y,z>0$ then in triangle ABC holds:

$$(x^2a^4+1)(y^2b^4+1)(z^2c^4+1)\ge 12(xy+yz+zx)\cdot F^2$$

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Applying Arkady Alt's inequality:

$(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$ **(with equality if and only if** $a=b=c=\frac{t}{\sqrt{2}}$**) and**

Oppenheim's inequality $a^2x+b^2y+c^2z\ge 4F\sqrt{xy+yz+zx}$**, it follows that**

$$(x^2a^4+1)(y^2b^4+1)(z^2c^4+1)\ge\frac{3}{4}(xa^2+yb^2+zc^2)^2\ge\frac{3}{4}\cdot 16F^2(xy+yz+zx)$$
$$=12(xy+yz+zx)\cdot F^2.$$

Equality holds if and only if $x=y=z=1$ **and** $a^2=b^2=c^2=\frac{1}{\sqrt{2}}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ **then the following relationship holds:**

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\ge$$

$$\ge\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.