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S.2562 In triangle ABC the following relationship holds:

$$(a^2r_a^2+1)(b^2r_b^2+1)(c^2r_c^2+1)\geq 27\cdot F^2$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2$ (with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$) it follows that

$$(a^2r_a^2+1)(b^2r_b^2+1)(c^2r_c^2+1)\geq\frac{3}{4}(ar_a+br_b+cr_c)^2\quad(1)$$

Since $r_a=\frac{F}{s-a}$ we obtain

$$ar_a+br_b+cr_c=F\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right)=$$

$$=\frac{F(s^2(a+b+c)-2s(ab+bc+ca)+3abc)}{(s-a)(s-c)(s-c)}=$$

$$=\frac{2Fs^2(s^2-s^2-r^2-4Rr+6Rr)}{s(s-a)(s-b)(s-c)}=\frac{2Fs^2r(2R-r)}{F^2}=2s(2R-r)\quad(2)$$

Using $(1),(2)$ and inequality $R\geq 2r$ (Euler), it results that

$$(a^2r_a^2+1)(b^2r_b^2+1)(c^2r_c^2+1)\geq\frac{3}{4}(ar_a+br_b+cr_c)^2=$$

$$=\frac{3}{4}\big(2s(2R-r)\big)^2\geq 3s^2(3r)^2=27\cdot F^2$$

Equality holds if and only if the triangle ABC is equilateral and $ar_a=\frac{1}{\sqrt{2}}$, that is $a=\frac{\sqrt{6}}{3}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.