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S.2569 If $x, y, z > 0$ then in triangle ABC holds:

$$\frac{a^3x^2}{r_a}+\frac{b^3y^2}{r_b}+\frac{c^3z^2}{r_c}\geq\frac{24\sqrt{3}\cdot(xy+yz+zx)r^3}{2R-r}$$

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Since $r_a=\frac{F}{s-a}$, we obtain

$$ar_a+br_b+cr_c=F\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right)=$$

$$=\frac{F(s^2(a+b+c)-2s(ab+bc+ca)+3abc)}{(s-a)(s-c)(s-c)}=$$

$$=\frac{2Fs^2(s^2-s^2-r^2-4Rr+6Rr)}{s(s-a)(s-b)(s-c)}=\frac{2Fs^2r(2R-r)}{F^2}=2s(2R-r)\quad(1)$$

Applying Bergström's inequality, Oppenheim's inequality

$a^2x+b^2y+c^2z\geq 4F\sqrt{xy+yz+zx}$, the relationship (1) and the inequality :

$s\geq 3\sqrt{3}r$ (item 5.11 from $[1]$), it follows that

$$\frac{a^3x^2}{r_a}+\frac{b^3y^2}{r_b}+\frac{c^3z^2}{r_c}=\frac{a^4x^2}{ar_a}+\frac{b^4y^2}{br_b}+\frac{c^4z^2}{rc_c}\geq\frac{(a^2x+b^2y+c^2z)^2}{ar_a+br_b+cr_c}\geq$$

$$\geq\frac{16F^2(xy+yz+zx)}{2s(2R-r)}=\frac{8sr^2(xy+yz+zx)}{2R-r}\geq\frac{24\sqrt{3}\cdot(xy+yz+zx)r^3}{2R-r}.$$

Equality holds if and only if the triangle ABC is equilateral and $x=y=z$.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969