ROMANIAN MATHEMATICAL MAGAZINE

S.2570 If $m>0$ and $a,b,c>0$, then:

$$(a^{2m+2}+2^{m+1})(b^{2m+2}+2^{m+1})(c^{2m+2}+2^{m+1})\ge\frac{2^{2m}}{3^{m-2}}(ab+bc+ca)^{m+1}$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that

$$(a^{2m+2}+2^{m+1})(b^{2m+2}+2^{m+1})(c^{2m+2}+2^{m+1})\ge$$

$$\ge\frac{3}{4}(2^{m+1})^2(a^{m+1}+b^{m+1}+c^{m+1})^2\ \ (1)$$

By Power mean inequality and the known inequality $a^2+b^2+c^2\ge ab+bc+ca$,

we obtain

$$\left(\frac{a^{m+1}+b^{m+1}+c^{m+1}}{3}\right)^{\frac{1}{m+1}}\ge\left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}\Rightarrow$$

$$3^{m-1}(a^{m+1}+b^{m+1}+c^{m+1})^2\ge(a^2+b^2+c^2)^{m+1}\ge(ab+bc+ca)^{m+1}\ \ (2)$$

Using (1) and (2), it results that

$$(a^{2m+2}+2^{m+1})(b^{2m+2}+2^{m+1})(c^{2m+2}+2^{m+1})\ge 3\cdot 2^{2m}\frac{1}{3^{m-1}}(ab+bc+ca)^{m+1}=$$

$$\ge\frac{2^{2m}}{3^{m-2}}(ab+bc+ca)^{m+1}.$$

Equality holds if and only if $a=b=c$ and $a^{m+1}=\sqrt{2^m}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.