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S.2572 In any triangle ABC the following relationship holds:

$$(a^2+t)(b^2+t)(c^2+t)\geq 9\sqrt{3}\cdot t^2F$$

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Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$ (with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$) and the inequality

$s^2\geq 3\sqrt{3}F$ (item 4.2 from $[1]$) it follows that

$$(a^2+t)(b^2+t)(c^2+t)\geq\frac{3}{4}\left(\sqrt{t}\right)^4(a+b+c)^2\geq\frac{3}{4}\cdot t^2\cdot 4\cdot 3\sqrt{3}F=9\sqrt{3}\cdot t^2F.$$

Equality if and only if the triangle ABC is equilateral and $a=b=c=\sqrt{\frac{t}{2}}$.

$[1]$ O. Bottema, *Geometric Inequalities,* Groningen 1969

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.