ROMANIAN MATHEMATICAL MAGAZINE

S.2573 If $t \geq 0$ then in any triangle ABC holds:

$$\frac{m_a^{t+1}}{\left(\sqrt{m_b m_c}\right)^{t+1}} \cdot \frac{a^{t+2}}{h_a^t} + \frac{m_b^{t+1}}{\left(\sqrt{m_c m_a}\right)^{t+1}} \cdot \frac{b^{t+2}}{h_b^t} + \frac{m_c^{t+1}}{\left(\sqrt{m_a m_b}\right)^{t+1}} \cdot \frac{c^{t+2}}{h_c^t} \geq 2^{t+2}\left(\sqrt{3}\right)^{1-t} \cdot F$$

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Solution by Titu Zvonaru-Romania

We have $ah_a = bh_b = ch_c = 2F$.

Applying $AM - GM$ and Carlitz's inequality $(abc)^{2/3} \geq \frac{4}{3}\sqrt{3}F$, it follows that

$$\frac{m_a^{t+1}}{\left(\sqrt{m_b m_c}\right)^{t+1}} \cdot \frac{a^{t+2}}{h_a^t} + \frac{m_b^{t+1}}{\left(\sqrt{m_c m_a}\right)^{t+1}} \cdot \frac{b^{t+2}}{h_b^t} + \frac{m_c^{t+1}}{\left(\sqrt{m_a m_b}\right)^{t+1}} \cdot \frac{c^{t+2}}{h_c^t} =$$

$$= \frac{m_a^{t+1}}{\left(\sqrt{m_b m_c}\right)^{t+1}} \cdot \frac{a^{2t+2}}{a^t h_a^t} + \frac{m_b^{t+1}}{\left(\sqrt{m_c m_a}\right)^{t+1}} \cdot \frac{b^{2t+2}}{b^t h_b^t} + \frac{m_c^{t+1}}{\left(\sqrt{m_a m_b}\right)^{t+1}} \cdot \frac{c^{2t+2}}{c^t h_c^t} \geq$$

$$\geq \frac{3}{(2F)^t}\left(\frac{m_a^{t+1}}{\left(\sqrt{m_b m_c}\right)^{t+1}} \cdot a^{2t+2} \cdot \frac{m_b^{t+1}}{\left(\sqrt{m_c m_a}\right)^{t+1}} \cdot b^{2t+2} \cdot \frac{m_c^{t+1}}{\left(\sqrt{m_a m_b}\right)^{t+1}} \cdot c^{2t+2}\right)^{\frac{1}{3}} =$$

$$= \frac{3}{2^t F^t}\left((abc)^{\frac{2}{3}}\right)^{t+1} \geq \frac{3}{2^t F^t}\left(\frac{4}{3}\sqrt{3}F\right)^{t+1} = 2^{t+2}\left(\sqrt{3}\right)^{1-t} \cdot F$$

Equality holds if and only if the triangle ABC equilateral.