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S.2576 In any triangle ABC the following relationship holds:

$$\left(\frac{1}{r_a^2}+s\right)\left(\frac{1}{r_b^2}+s\right)\left(\frac{1}{r_c^2}+s\right)\geq\frac{81}{4}$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\leq\frac{3}{4}t^4(a+b+c)^2$

(with equality in and only if $a=b=c=\frac{t}{\sqrt{2}}$) it follows that

$$\left(\frac{1}{r_a^2}+s\right)\left(\frac{1}{r_b^2}+s\right)\left(\frac{1}{r_c^2}+s\right)\geq\frac{3}{4}(\sqrt{s})^4\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^2 \quad (1)$$

Since $r_a=\frac{F}{s-a}$ we obtain

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{s-a}{F}+\frac{s-b}{F}+\frac{s-c}{F}=\frac{s}{sr}=\frac{1}{r} \quad (2)$$

Using $(1),(2)$ and the inequality $s^2\geq 27r^2$ (item 5.11 from $[1]$), it results that

$$\left(\frac{1}{r_a^2}+s\right)\left(\frac{1}{r_b^2}+s\right)\left(\frac{1}{r_c^2}+s\right)\geq\frac{3}{4}s^2\cdot\frac{1}{r^2}\geq\frac{3}{4}\cdot 27r^2\cdot\frac{1}{r^2}=\frac{81}{4}.$$

Equality holds if and only if the triangle ABC is equilateral.

[1] O. Bottema, *Geometric Inequalities*, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.