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S.2597 In any triangle ABC the following relationship holds:

$$\left(\left(\frac{a^3}{bR+cr}\right)^2+1\right)\left(\left(\frac{b^3}{cR+ar}\right)^2+1\right)\left(\left(\frac{c^3}{aR+br}\right)^2+1\right)\ge\frac{36}{(R+r)^2}\cdot F^2$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), Bergström's inequality, the known inequality $ab+bc+ca\le a^2+b^2+c^2$ and Ionescu-Weitzenbock's inequality:

$a^2+b^2+c^2\ge 4\sqrt{3}F$, it follows that:

$$\left(\left(\frac{a^3}{bR+cr}\right)^2+1\right)\left(\left(\frac{b^3}{cR+ar}\right)^2+1\right)\left(\left(\frac{c^3}{aR+br}\right)^2+1\right)\ge$$

$$\ge\frac{3}{4}\left(\frac{a^3}{bR+cr}+\frac{b^3}{cR+ar}+\frac{c^3}{aR+br}\right)^2=$$

$$=\frac{3}{4}\left(\frac{a^4}{abR+car}+\frac{b^4}{bcR+abr}+\frac{c^4}{caR+bcr}\right)^2\ge$$

$$\ge\frac{3}{4}\cdot\frac{(a^2+b^2+c^2)^4}{\left((R+r)(ab+bc+ca)\right)^2}\ge\frac{3(a^2+b^2+c^2)^2}{4(R+r)^2}\ge\frac{3\left(4\sqrt{3}F\right)^2}{4(R+r)^2}=\frac{36}{(R+r)^2}\cdot F^2.$$

Equality holds if and only if the triangle ABC is equilateral

and $\frac{a^3}{aR+ar}=\frac{1}{\sqrt{2}}$, that is $a=\sqrt{6}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\ge\frac{3}{4}t^2\left((x+y)^2+t^2\right)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.