ROMANIAN MATHEMATICAL MAGAZINE

S.2600 If $x \geq 0$, then in the triangle ABC the following relationship holds:

$$\left(\frac{a^{2x+4}}{(bR+cr)^{2x}}+2\right)\left(\frac{b^{2x+4}}{(cR+ar)^{2x}}+2\right)\left(\frac{c^{2x+4}}{(aR+br)^{2x}}+2\right) \geq \frac{144}{(R+r)^{2x}} \cdot F^2$$

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Applying Arkady Alt's $(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^2$

(with the equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), Radon's inequality, the known inequality $ab+bc+ca \leq a^2+b^2+c^2$ and Ionescu-Weitzenbock's inequality

$a^2+b^2+c^2 \geq 4\sqrt{3}F$, it follows that:

$$\left(\frac{a^{2x+4}}{(bR+cr)^{2x}}+2\right)\left(\frac{b^{2x+4}}{(cR+ar)^{2x}}+2\right)\left(\frac{c^{2x+4}}{(aR+br)^{2x}}+2\right) \geq$$

$$\geq \frac{3}{4}(\sqrt{2})^2\left(\frac{a^{x+2}}{(bR+cr)^x}+\frac{b^{x+2}}{(cR+ar)^x}+\frac{c^{x+2}}{(aR+ar)^x}\right)^2 =$$

$$= 3\left(\frac{(a^2)^{x+1}}{(abR+car)^x}+\frac{(b^2)^{x+1}}{(bcR+abr)^x}+\frac{(c^2)^{x+1}}{(aR+ar)^x}\right)^2 \geq 3\left(\frac{(a^2+b^2+c^2)^{x+1}}{(R+r)^x(ab+bc+ca)^x}\right)^2$$

$$\geq \frac{3(a^2+b^2+c^2)^2}{(R+r)^{2x}} \geq \frac{3\left(4\sqrt{3}F\right)^2}{(R+r)^{2x}} = \frac{144}{(R+r)^{2x}} \cdot F^2.$$

Equality holds if and only if the triangle ABC is equilateral and $\frac{a^{x+2}}{(aR+ar)^x}=1$, that is $a=\frac{2\sqrt{3}}{3}$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.