ROMANIAN MATHEMATICAL MAGAZINE

**S.2604** If $t,a,b,c,x,y>0$ then:

$$\left(\frac{a^2}{(bx+cy)^2}+t^2\right)\left(\frac{b^2}{(cx+ay)^2}+t^2\right)\left(\frac{c^2}{(ax+by)^2}+t^2\right)\ge\frac{27t^4}{4(x+y)^2}$$

*Proposed by D.M.Bătinețu-Giurgiu, Gheorghe Boroica – Romania*

*Solution by Titu Zvonaru-Romania*

Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that:

$$\left(\frac{a^2}{(bx+cy)^2}+t^2\right)\left(\frac{b^2}{(cx+ay)^2}+t^2\right)\left(\frac{c^2}{(ax+by)^2}+t^2\right)\ge$$

$$\ge\frac{3}{4}t^4\left(\frac{a}{bx+cy}+\frac{b}{cx+ay}+\frac{c}{ax+by}\right)^2\quad(1)$$

By Bergström's inequality and the known inequality:

$(a+b+c)^2\ge3(ab+bc+ca)$ we get

$$\frac{a}{bx+cy}+\frac{b}{cx+ay}+\frac{c}{ax+by}=\frac{a^2}{abx+cay}+\frac{b^2}{bcx+aby}+\frac{c^2}{cax+bcy}\ge$$

$$\ge\frac{(a+b+c)^2}{x(ab+bc+ca)+y(ab+bc+ca)}\ge\frac{3(ab+bc+ca)}{(x+y)(a+b+c)}\quad(2)$$

Using $(1)$ and $(2)$ we obtain:

$$\left(\frac{a^2}{(bx+cy)^2}+t^2\right)\left(\frac{b^2}{(cx+ay)^2}+t^2\right)\left(\frac{c^2}{(ax+by)^2}+t^2\right)\ge\frac{3}{4}t^4\left(\frac{3}{x+y}\right)^2=\frac{27t^4}{4(x+y)^2}$$

Equality holds if and only if $a=b=c$ and $x+y=\frac{\sqrt{2}}{t}$.

**ARKADY ALT'S INEQUALITY**

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

**with equality if and only if** $x = y = z = \frac{t}{\sqrt{2}}$.

**Proof: We have**

$$(x^2 + t^2)(y^2 + t^2) \geq \frac{3}{4} t^2((x + y)^2 + t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x - y)^2 \geq 0.$$

**Applying Cauchy-Buniakovski-Schwarz inequality we obtain**

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3t^2}{4}((x + y)^2 + t^2)(t^2 + z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x + y) + tz)^2 = \frac{3}{4} t^4 (x + y + z)^2.$$

**The equality holds if and only if** $x = y = z = \frac{t}{\sqrt{2}}$.