S.2620 In any triangle ABC with the area F the following relationship holds:

$$(a^2+2)^3+(b^2+2)^3+(c^2+2)^3\geq 108\sqrt{3}F$$

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Applying $AM-GM$ inequality, Arkady Alt's inequality

$(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq \frac{3}{4}t^4(a+b+c)^2$ (with equality if and only if

$a=b=c=\frac{t}{\sqrt{2}}$), Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\geq 4\sqrt{3}F$ and

Gordon's inequality $ab+bc+ca\geq 4\sqrt{3}F$, it follows that

$$(a^2+2)^3+(b^2+2)^3+(c^2+2)^3\geq 3(a^2+2)(b^2+2)(c^2+2)\geq$$

$$\geq 3\cdot\frac{3}{4}\left(\sqrt{2}\right)^4(a+b+c)^2=$$

$$=9\left(a^2+b^2+c^2+2(ab+bc+ca)\right)\geq 9\left(12\sqrt{3}F\right)=108\sqrt{3}F.$$

Equality holds if and only if $a=b=c=1$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.