ROMANIAN MATHEMATICAL MAGAZINE

S.2624 If $m,n,x,y,z>0$ then:

$$\left(\frac{(mx+ny)^2}{z^2}+2\right)\left(\frac{(my+nz)^2}{x^2}+2\right)\left(\frac{(mz+nx)^2}{y^2}+2\right)\geq 27(m+n)^2$$

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Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$) and $AM-GM$ inequality, it follows that

$$\left(\frac{(mx+ny)^2}{z^2}+2\right)\left(\frac{(my+nz)^2}{x^2}+2\right)\left(\frac{(mz+nx)^2}{y^2}+2\right)=$$

$$\geq \frac{3}{4}(\sqrt{2})^2\left(\frac{mx+ny}{z}+\frac{my+nz}{x}+\frac{mz+nx}{y}\right)^2=$$

$$=3\left(m\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)+n\left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right)\right)^2\geq 3(3m+3n)^2=27(m+n)^2.$$

Equality holds if and only if $x=y=z$ and $\frac{mx+ny}{z}=\frac{my+nx}{x}=\frac{mz+nx}{y}=1$,

that is $x=y=z=1, m+n=1$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq \frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.