S.2627 In any triangle ABC the following relationship holds:

$$\left(\frac{1}{a^2b^2}+2\right)\left(\frac{1}{b^2c^2}+2\right)\left(\frac{1}{c^2a^2}+2\right)\geq\frac{3}{R^4}$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$)

and the inequality $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\geq\frac{1}{R^2}$ (item 5.24 from $[1]$), it follows that

$$\left(\frac{1}{a^2b^2}+2\right)\left(\frac{1}{b^2c^2}+2\right)\left(\frac{1}{c^2a^2}+2\right)\geq\frac{3}{4}(\sqrt{2})^4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2\geq\frac{3}{R^4}.$$

Equality holds if and only if the triangle ABC is equilateral and

$\frac{1}{ab}=\frac{1}{bc}=\frac{1}{ca}=\frac{\sqrt{2}}{\sqrt{2}}$, that is $a=b=c=1$.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.