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S.2628 If M is an interior point of the triangle ABC and d_a, d_b, d_c are the distances of point M to the sides BC, CA respectively AB, then:

$$\left(\frac{a^2}{d_a^2}+2\right)\left(\frac{b^2}{d_b^2}+2\right)\left(\frac{c^2}{d_c^2}+2\right)>324$$

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We have $ad_a+bd_b+cd_c=2F$. Applying Arkady Alt's inequality

$(a^2+t^2)(b^2+t^2)(c^2+t^2)\ge\frac{3}{4}t^4(a+b+c)^2$, Bergström's inequality it follows that

$$\left(\frac{a^2}{d_a^2}+2\right)\left(\frac{b^2}{d_b^2}+2\right)\left(\frac{c^2}{d_c^2}+2\right)\ge\frac{3}{4}(\sqrt{2})^4\left(\frac{a}{d_a}+\frac{b}{d_b}+\frac{c}{d_c}\right)^2=3\left(\frac{a^2}{ad_a}+\frac{b^2}{bd_b}+\frac{c^2}{cd_c}\right)^2\ge$$

$$\ge 3\left(\frac{(a+b+c)^2}{ad_a+bd_b+cd_c}\right)^2=\frac{3(a+b+c)^4}{4F^2}\qquad(1)$$

Using Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\ge 4\sqrt{3}F$ and Gordon's inequality

$ab+bc+ca\ge 4\sqrt{3}F$, we obtain

$$(a+b+c)^4=\left(a^2+b^2+c^2+2(ab+bc+ca)\right)^2\ge\left(4\sqrt{3}F+8\sqrt{3}F\right)^2=432F^2\quad(2)$$

By (1) and (2) yields

$$\left(\frac{a^2}{d_a^2}+2\right)\left(\frac{b^2}{d_b^2}+2\right)\left(\frac{c^2}{d_c^2}+2\right)\ge\frac{3(432F^2)}{4F^2}=324.$$

Equality holds if and only if the triangle ABC is equilateral and $\frac{a}{d_a}=\frac{\sqrt{2}}{\sqrt{2}}$, which is false.

Yields that the inequality is strict.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.