ROMANIAN MATHEMATICAL MAGAZINE

S.2629 If $x, y, z > 0$, then:

$$\left(\frac{(x+y)^2}{z^2}+2\right)\left(\frac{(y+z)^2}{x^2}+2\right)\left(\frac{(z+x)^2}{y^2}+2\right)\geq 108$$

Proposed by D.M.Bătinețu-Giurgiu, Dorina Goiceanu – Romania

Solution by Titu Zvonaru-Romania

We will use the following inequalities:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq \frac{3}{4}t^4(a+b+c)^2 \qquad (1)$$

(Arkady Alt's inequality, with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$)

and $\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}\geq 6 \Leftrightarrow \frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\geq 6 \qquad (2)$

It follows that

$$\left(\frac{(x+y)^2}{z^2}+2\right)\left(\frac{(y+z)^2}{x^2}+2\right)\left(\frac{(z+x)^2}{y^2}+2\right)\geq$$

$$\geq \frac{3}{4}\left(\sqrt{2}\right)^4\left(\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}\right)^2\geq 3(6)^2=108.$$

Equality holds if and only if $x=y=z$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq \frac{3}{4}t^2\left((x+y)^2+t^2\right) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq \frac{3t^2}{4}\left((x+y)^2+t^2\right)(t^2+z^2)\geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.