ROMANIAN MATHEMATICAL MAGAZINE

S2630 If $x,y,z>0$, then:

$$\left(\frac{x^2}{(y+z)^2}+2\right)\left(\frac{y^2}{(z+x)^2}+2\right)\left(\frac{z^2}{(x+y)^2}+2\right)>\frac{27}{4}$$

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We will use the following inequalities:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2 \qquad (1)$$

(Alt's inequality, with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$)

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2} \qquad (2)$$

(Nesbitt's inequality, with equality if and only if $x=y=z$).

It follows that

$$\left(\frac{x^2}{(y+z)^2}+2\right)\left(\frac{y^2}{(z+x)^2}+2\right)\left(\frac{z^2}{(x+y)^2}+2\right)\geq$$

$$\geq\frac{3}{4}(\sqrt{2})^4\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2\geq 3\left(\frac{3}{2}\right)^2=\frac{27}{4}.$$

Since for $x=y=z$ we have

$$\left(\frac{x^2}{(y+z)^2}+2\right)\left(\frac{y^2}{(z+x)^2}+2\right)\left(\frac{z^2}{(x+y)^2}+2\right)=\left(\frac{9}{4}\right)^3>\frac{27}{4},$$

the inequality is strict.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.